Gaussian Process Models for Mortality Improvement Factors

Longevity 12

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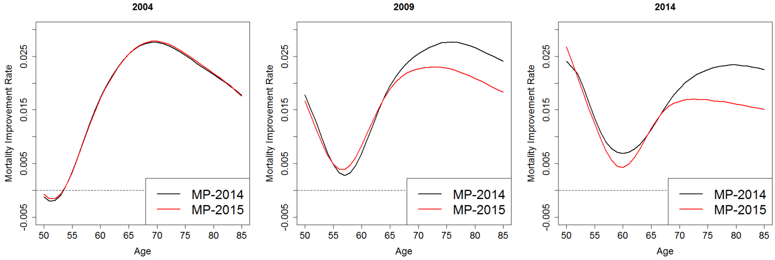
US Mortality Improvement

MP-2014 / MP-2015

- Published by SOA
- US CDC Data
- MP-2014 uses years 1950-2009
- Plans to update scales at least triennially; two years of additional CDC data shows drastic change in later years
  - MP-2015 emerges



Goal:

Model US Mortality data using Gaussian Process (GP) regression

- Bayesian
- Provides posterior Gaussian distribution for input of any age and year
- Offers easy analysis of both mortality and mortality improvement simultaneously
- Gaussian distribution implies one-year mortality improvement factors remains Gaussian
- Differentiable: can provide instantaneous mortality improvement (still Gaussian)
- Spatial approach inherently handles missing and edge data

Typical Regression Assumption

Hypothesis:

$$y = f(x) + \varepsilon$$

- Observe $\boldsymbol{y} = y^{1:N}$ for input locations $\boldsymbol{x} = x^{1:N}$
- Want to understand the function f
  - e.g. $f(x) = \beta_0 + \beta_1 x$ (simple linear regression)
- ε is noise: can't observe $f(x)$ directly
- Assume $\varepsilon \sim \mathcal{N}\left(0, \sigma^2(x)\right)$ (often $\sigma(x) \equiv \sigma \in \mathbb{R}^+$)
- Our assumption: f is a Gaussian Process (modeling log-mortality)

Gaussian Process

- Defined as a set of random variables $\{f(x) | x \in \mathbb{R}^d\}$
- Any finite subset has a multivariate Gaussian distribution with covariance $C(\cdot, \cdot)$:

$$f(x_1), \ldots, f(x_n) \sim \mathcal{N}\left((m(x_1), \ldots, m(x_n)), C(\boldsymbol{x}, \boldsymbol{x}^T)\right).$$

- Fix mean function m and covariance kernel C; this provides a prior distribution

Posterior

- Observe pairs $(\boldsymbol{y}, \boldsymbol{x}) = ((y, x)^{1:N})$
  - (e.g. $y =$ historic log–mortality and $x =$ (age, year))
- Gaussian assumptions imply that marginally for any input x

$$f(x)|(\boldsymbol{y}, \boldsymbol{x}) \sim \mathcal{N}\left(m_*(x), s_*^2(x)\right)$$

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$$f(x)|(\boldsymbol{y}, \boldsymbol{x}) \sim \mathcal{N}\left(m_*(x), s_*^2(x)\right)$$

- m_* and s_*^2 are the posterior mean and variance functions

$$\begin{cases} m_*(x) \doteq \boldsymbol{c}(x)^T(\boldsymbol{C} + \Sigma)^{-1}\boldsymbol{y}; \\ s_*^2(x) \doteq C(x, x) - \boldsymbol{c}(x)^T(\boldsymbol{C} + \Sigma)^{-1}\boldsymbol{c}(x), \end{cases} \tag{1}$$

where

$$\begin{cases} \boldsymbol{c}(x) \doteq \left(C(x, x^i)\right)_{1 \le i \le N} & \text{(covariances between } x \text{ and inputs } \boldsymbol{x}) \\ \boldsymbol{C} \doteq \left(C(x^i, x^j)\right)_{1 \le i,j \le N} & \text{(covariances between inputs } \boldsymbol{x}) \\ \Sigma \doteq \operatorname{diag}\left(\sigma^2(x^i)\right) & \text{(diagonal matrix of noise variance)} \end{cases}$$

Covariance Kernels & Parameter Estimation

- Common choice is squared-exponential (or Gaussian) covariance kernel

$$C(x, x') = \eta^2 \exp\left(-\frac{(x_{ag} - x'_{ag})^2}{2\theta_{ag}^2} - \frac{(x_{yr} - x'_{yr})^2}{2\theta_{yr}^2} \right).$$

- Knowing mortality at x will greatly influence mortality at “neighboring" x's
  - e.g. knowing mortality for a 80 year old in 2015 greatly aids in prediction of a 85 year old's mortality in 2016; knowing a 50 year old's mortality in 2000 has a nearly non-existent effect
- Implies hyperparameter family of $\Theta \doteq (\theta_{ag}, \theta_{yr}, \eta^2, \sigma^2)$
  - Also mean function hyperparameters (if included)
- Estimates are fit using MLE; likelihood can be written out explicitly due to Gaussian assumptions
  - Done using `R` package `DiceKriging`
- Alternatively, can use Bayesian approach with priors on Θ
  - Separate package using STAN language
  - Leads to non-Gaussian posterior

Illustrative Example

Goal: Learn $f(x) = \sin(x)$ over domain $[0, 2\pi]$

- Observe realizations of

$$y = \sin(x) + \varepsilon$$

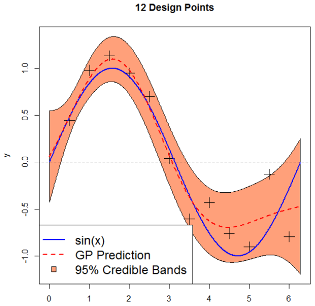
  where $\varepsilon \sim N(0, 0.01x)$
- Try:
  - $x = 0.5, 1, 1.5, \ldots, 5.5, 6;$
  - $x = 0.25, 0.5, 0.75, \ldots, 5.75, 6$

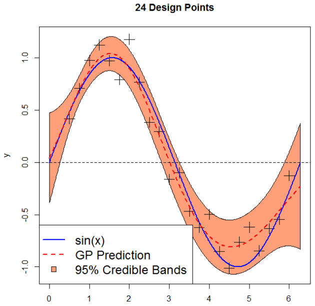
Illustrative Example

$$y = \sin(x) + \varepsilon, \qquad \varepsilon \sim \mathcal{N}(0, 0.01x)$$

1. Uncertainty increases at edges (especially for large x: $\sigma(x) = 0.01x$)
2. Uncertainty decreases as $N = 12 \rightarrow N = 24$
3. Accuracy of fit increases as $N = 12 \rightarrow N = 24$

$x = 0.5, 1, 1.5, \ldots, 5.5, 6$ (12 points)	$x = 0.25, 0.5, 0.75, \ldots, 5.75, 6$ (24 points)





Data

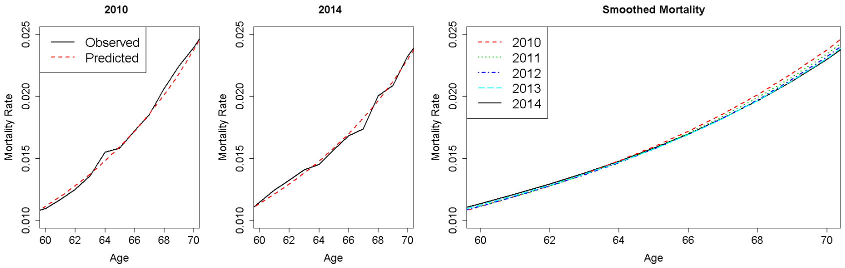
- CDC Data
  - United States
  - Ages 50–84, Years 1999–2014
    - ★ 1360 Data Points ($x = (x_{ag}, x_{yr})$)
    - ★ 84 is maximal age for CDC data
    - ★ 50 chosen as cutoff to minimize mixing lower age behavior
    - ★ 1999 earliest year available on wonder.cdc.gov
    - ★ Could add earlier years, but our analysis suggests they have little effect
    - ★ Most relevant for longevity risk

Model Assumptions

- Observe central mortality rate $e^{-\mu(x_{ag}, x_{yr})} = D(x_{ag,yr})/E(x_{ag,yr})$
- Fit log-mortality rate y to $x = x_{ag}, x_{yr}$ pairs
- Can try $\sigma(x)$ based on Binomial assumption
  - Overdispersion issues ($\mu_{ag,yr}$ is unknown)
  - Minimal change in final model from simply choosing $\sigma := \sigma(x)$
- Use Gaussian covariance kernel
  - Implies f is differentiable
  - Minimal change in final model from other kernel choices

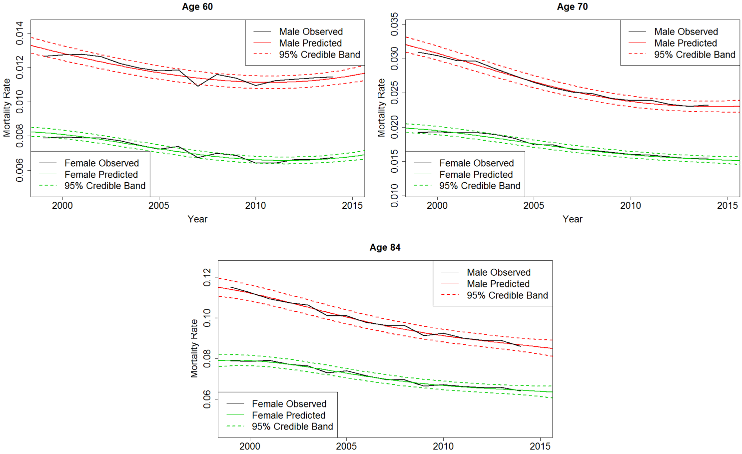
Posterior Predicted Mortality Rates

- Showing $m_*(x)$ for each ages 60–70
- Left panels include historic observations
- Right panel suggests mortality improvement



Mortality Over Time with Credible Bands

- Posterior mean and 95% credibility bands for f_* over calendar year
- Can observe increasing uncertainty at edges
- Observe mortality improvement then decline



Mortality Improvement

- Typical way is to look at the annual backward improvement

$$MI^{obs}_{back}\left(x_{ag}; yr\right) \doteq 1 - \frac{\exp\left(\mu(x_{ag}, yr)\right)}{\exp\left(\mu(x_{ag}, yr-1)\right)}$$

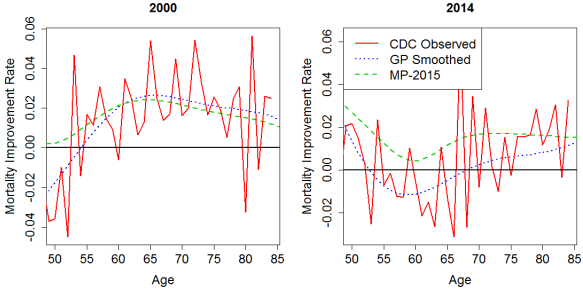
- $f_*(x_{ag}, yr)$ is a random variable, so we have the predicted mean improvement

$$m^{GP}_{back}\left(x_{ag}, yr\right) = \mathbb{E}\left[MI^{GP}_{back}\left(x_{ag}, yr\right)\right] \doteq \mathbb{E}\left[1 - \frac{\exp\left(f_*(x_{ag}, yr)\right)}{\exp\left(f_*(x_{ag}, yr-1)\right)}\right]$$

  - Available in closed form (lognormal distribution)
- Also have $MI^{MP}_{back}\left(x_{ag}; yr\right)$ (published MP-2015 improvement factors)

Comparing Mortality Improvement Methods

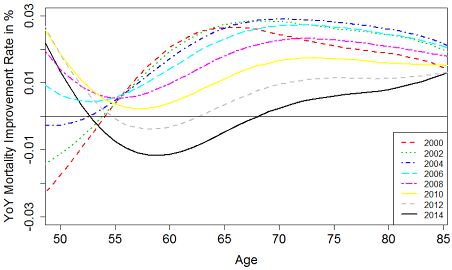
- Raw improvements extremely noisy (unsurprising)
- Smoothed methods both follow data well
- GP implies a stronger decline
  - Additional data suggests mortality deceleration



GP Improvement Over Time

GP Improvements from 2000–2014 (in 2 year increments)

- Shape changes (flips) over time
- Consistent with MP-2015
- Generally decelerating after age 55



Backward Difference & Derivatives

$$1 - \left(\frac{\exp\left(f_*(x_{ag}, yr)\right)}{\exp\left(f_*(x_{ag}, yr - h)\right)} \right)^{1/h} \approx -\frac{f_*(x_{ag}, yr) - f_*(x_{ag}, yr - h)}{h} \qquad (3)$$

- As defined, the annual mortality improvements are backward differences with $h = 1$
- Right side remains a GP by linearity
- Taking limit as $h \to 0$ yields derivative
  - Exists (depending on covariance kernel)
- Closed form expressions for distribution of $\frac{\partial f_*}{\partial x_{yr}}$

GP Derivative

Proposition

For the Gaussian Process f_* with a twice differentiable covariance kernel C, the limiting random variables

$$\frac{\partial f_*}{\partial x_{yr}}(x_{ag}, yr) \doteq \lim_{h \to 0} \frac{f_*(x_{ag}, yr+h) - f_*(x_{ag}, yr)}{h} \tag{4}$$

exist in mean square and form a Gaussian process $\frac{\partial f_*}{\partial x_{yr}} \sim GP(m_{diff}, s_{diff})$. Given the training set $\mathcal{D} = (\mathbf{x}, \mathbf{y})$, the posterior distribution of $\frac{\partial f_*}{\partial x_{yr}}(x_*)$ has mean and variance

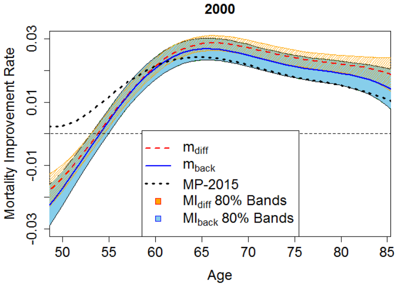
$$\begin{cases} m_{diff}(x_*) = \mathbb{E}\left[\frac{\partial f_*}{\partial x_{yr}}(x_*) \middle| \mathbf{x}, \mathbf{y}\right] = \frac{\partial C}{\partial x'_{yr}}(\mathbf{x}, x_*)(\mathbf{C} + \Sigma)^{-1}\mathbf{y}, \\ s^2_{diff}(x_*) = Var\left(\frac{\partial f_*}{\partial x_{yr}}(x_*) \middle| \mathbf{x}, \mathbf{y}\right) = \frac{\partial^2 C}{\partial x_{yr}\partial x'_{yr}}(x_*, x_*) - \frac{\partial C}{\partial x'_{yr}}(\mathbf{x}, x_*)(\mathbf{C} + \Sigma)^{-1}\frac{\partial C}{\partial x_{yr}}(x_*, \mathbf{x}), \end{cases}$$

where $\frac{\partial C}{\partial x'_{yr}}(\mathbf{x}, x_*) = \left[\frac{\partial C}{\partial x'_{yr}}(x^1, x_*), \ldots, \frac{\partial C}{\partial x'_{yr}}(x^N, x_*)\right]$ and each component is computed as the partial derivative of $C(x, x')$.

See Theorem 2.2.2 in Adler (2010) for more details/proof.

Comparing Other Methods with GP Derivative

- Blue is backwards mortality difference (as before); red is GP derivative; black is MP-2015
- Analysis of other years shows deceleration begins around 2010
  - Implies mortality evolution is convex
    - Justifies accelerating divergence between yearly difference and derivative methods
  - MP-2014 and MP-2015 begin to diverge around 2010
    - Suggests that later years are crucial to mortality forecasts



Conclusions

- GP's provide a variety of benefits to modeling mortality and mortality improvement
  - Bayesian approach (data driven)
  - Posterior distribution for any location
    - Including distribution of mortality improvement (both yearly difference and instantaneous)
    - Credible bands (historic and forecasting)
- Relatively consistent results with MP-2015
  - Four years of additional data pushes GP results in the direction that MP-2015 took compared to MP-2014
  - Differences in results is likely due to data differences than model issues
- GP framework easily handles joint analysis of mortality rates and mortality improvement

Future Work

- Modeling annual mortality improvement directly with GP
- Monotonicity constraint
- Multiple populations
  - Jointly modeling male & female mortality
  - Multivariate GP of multiple countries
- Modeling by cause of death
- More detailed backtesting
  - Analysis on other countries (individually)

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THANK YOU!

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